ON THE EXISTENCE OF SIMPLEX SPACES

BY
ALAN GLEIT*

ABSTRACT
We give several general theorems to facilitate the construction of examples of simplex spaces.

Edward Effros in [4] defined a simplex space to be an ordered Banach space with closed positive cone whose dual is a Kakutani L-space. Much is known about the structure of such spaces [4, 5, 6, 7, 8, 9, 12]. However, there is a distinct lack of many interesting examples. This paper provides the first general results which may help to alleviate this problem.

We give the existence theorems in Section 1. The major result is Theorem 1.2 which contains the following important example as a special case. Let Y be a compact Hausdorff space and $X = \{x_j \mid j = 1,2,\cdots\} \cup \{z\} \subseteq Y$ be a sequence of distinct points such that $\lim x_j = z$. Let $\{\mu_j \mid j = 1,2,\cdots\}$ and ν be measures on Y with norms bounded by one satisfying:

1) $\mu_j(X) = \nu(X) = 0 \qquad j = 1,2,\cdots$
2) $\mu_j \to \nu$ in the weak* topology.

If $X \neq Y$, then

$$V = \{f \in C(Y) \mid f(z) = \nu(f), f(x_j) = \mu_j(f), j = 1,2,\cdots\}$$

is a non-trivial simplex space with the relative norm and order. Section 2 is devoted to two examples.

1. The Existence Theorems

We recall, first, that we call a Banach lattice B a (Kakutani) *L-space* if it satisfies

1) $\|x\| = \||x|\|$ for all $x \in B$ and
2) for all positive $x, y \in B$, $\|x + y\| = \|x\| + \|y\|$.

* An N.S.F. Postdoctoral Fellow supported by N.S.F. development grant GU 2056.

Received December 17,1969 and in revised form July 9, 1970

If Y is a compact Hausdorff space, we let $C(Y)$ be the space of (real) continuous functions on Y with the natural pointwise order and the supremum norm. Its dual, $C^*(Y)$, is the space of all regular Borel measures on Y [3, IV. 6.3]. If $\mu \in C^*(Y)$, the norm of μ is the total variation of μ over Y, i.e.

$$\begin{aligned} \|\mu\| &= |\mu|(Y) \\ &= (\mu^+ + \mu^-)(Y). \end{aligned}$$

It is clear that $C^*(Y)$ is an L-space.

If X is a Borel subset of Y, we let $C^*(Y;X)$ be the space of all regular Borel measures on Y whose total variation on X equals zero. If $y \in Y$, we let $\delta(y)$ be the point mass at y, i.e. $\delta(y)$ is that element of $C^*(Y)$ which satisfies

$$\delta(y)(f) = f(y)$$

for each $f \in C(Y)$.

PROPOSITION 1.1. *Let Y be a compact Hausdorff space and X a Borel subset of Y. Then $C^*(Y;X)$ is an L-space. Further, the extreme points of the positive part of the unit ball of $C^*(Y;X)$ are precisely*

$$\{\delta(y) \mid y \in Y - X\} \bigcup \{0\}.$$

PROOF. $C^*(Y;X)$ is obviously a norm closed subspace of $C^*(Y)$. As $\mu \in C^*(Y;X)$ iff $|\mu| \in C^*(Y;X)$, $C^*(Y;X)$ is a lattice and so is an L-space. Let E be the positive elements in the unit ball of $C^*(Y;X)$ and suppose $\mu \in E$. If $\mu = \frac{1}{2}(\mu_1 + \mu_2)$ with μ_i in the positive unit ball of $C^*(Y)$, one easily gets $\mu_1(X) = \mu_2(X) = 0$ and so $\mu_i \in E$. Thus, E is a face of the positive unit ball of $C^*(Y)$. Hence for $\mu \in E$, μ is extreme in E iff μ is extreme in the positive unit ball of $C^*(Y)$. As the latter is $\{\delta(y) \mid y \in Y\} \cup \{0\}$, the proposition is clear.

We call an ordered Banach space V with closed positive cone a *simplex space* if V^* is an L-space [4]. We let

$$P_1(V) = \{f \in V^* \mid \|f\| \leqq 1, \ f(x) \geqq 0 \ \text{for each} \ x \geqq 0\}$$

and $EP_1(V)$ be its extreme points. We are now prepared for our main theorem.

THEOREM 1.2. *Let Y be a compact Hausdorff space with X a closed subset. Let $x \to \mu_x$ be a weak* continuous map of X into the positive part of the unit ball of $C^*(Y)$. Let $X = X_1 \bigcup X_2$ where*

$$X_2 = \{x \in X \mid \mu_x = \delta(x)\}.$$

We assume:

1) *For each* $x \in X_1$, $\mu_x(X_1) = 0$.

2) $X_1 \neq Y$.

Then

$$V = \{f \in C(Y) \mid f(x) = \mu_x(f) \text{ for all } x \in X\}$$

is a non-trivial simplex space with the relative order and norm inherited from $C(Y)$. *Further,* V^* *is isometrically order isomorphic to* $C^*(Y; X_1)$ *and so*

$$EP_1(V) = \{\delta(y) \mid y \in Y - X_1\} \bigcup \{0\}.$$

Further, if Y *is a metric space, then* V *is separable.*

PROOF. We first note that V is closed in $C(Y)$ and so is a Banach space. Also, it obviously has a closed positive cone. Using Proposition 1.1, we have only to show that $V \neq \{0\}$ and that V^* is isometrically order isomorphic to $C^*(Y; X_1)$.

We first show $V \neq \{0\}$. We assume $X_2 \neq \emptyset$ with no loss of generality (if it is empty, add a point from $Y - X$ to X). We then let

$$X'_1 = \{\mu_x - \delta(x) \mid x \in X_1\}$$

and

$$X' = X'_1 \bigcup \{0\} = \{\mu_x - \delta(x) \mid x \in X\}.$$

As the map

$$x \to \mu_x - \delta(x) \tag{1.1}$$

is continuous, X' is weak* compact. Obviously, the map (1.1) is one-to-one from X_1 onto X'_1. Since X_2 is closed (it is the inverse image of a point), both X_1 and X'_1 are locally compact. As the map (1.1) is continuous from X to X', it is a proper map [2, Ch. 1, §10, no. 3, Cor.]. Hence X_1 is homeomorphic to X'_1 [2, Ch. 1, §10, no. 1, Prop. 2].

Let Z be the weak* closed convex hull of X'. Then [11, Prop. 1.2] implies that

$Z = \{z \in C^*(Y) \mid$ There exists a regular Borel probability measure ν' supported by X' with resultant $z\}$.

Let W be the linear span of Z. Then obviously

$W = \{w \in C^*(Y) \mid$ There exists a bounded regular Borel measure ν' supported by X' for which $w(f) = \int_{X'} p(f) d\nu'(p)$ for each $f \in C(Y)\}$.

Let $w \in W$. Then $w(f) = \int_{X'} p(f) d\nu'(p)$ for some measure ν' on X'. We shall write $w = r(\nu')$ for this. We now associate with ν' a measure ν on X_1 by the homeomorphism between it and X_1'. In more detail, if A is a Borel set in X_1, we let $A' = \{\mu_x - \delta(x) \mid x \in A\}$, which is a Borel set in X_1'. We then define

$$v(A) \equiv v'(A'). \tag{1.2}$$

One easily verifies that v is a regular Borel measure on X_1.

We explicitly remove the possible atom at 0 from v'; call

$$\mu' = v' - v'(\{0\})\delta(0).$$

Let $f \in C(Y)$. Then

$$\begin{aligned} w(f) &= r(v')(f) \\ &= \int_{X'} p(f)dv'(p) \\ &= \int_{X_1'} p(f)d\mu'(p) \\ &= r(\mu')(f) \end{aligned}$$

and so $w = r(\mu')$. Writing μ' as the limit of finite sums of point measures

$$\mu' = \lim_\alpha \sum_i \lambda_{\alpha,i}\delta(p_{\alpha,i})$$

where $p_{\alpha,i} = \mu_{x_{\alpha,i}} - \delta(x_{\alpha,i})$, we see that

$$v = \lim_\alpha \sum_i \lambda_{\alpha,i}\delta(x_{\alpha,i}).$$

By going to these sums and taking the limits, we get, for each $f \in C(Y)$,

$$\begin{aligned} w(f) &= \int_{X_1'} p(f)d\mu'(p) \\ &= \int_{X_1} (\mu_x(f) - \delta(x)(f))dv(x) \\ &= \int_{X_1} \mu_x(f)dv(x) - v(f). \end{aligned}$$

As the map $x \to \mu_x$ is continuous and X_1 is locally compact, [10, V, §3, Cor. to. Prop. 12] yields

$$w(h) = \int_{X_1} \mu_x(h)dv(x) - v(h)$$

for each positive Borel measurable h. In particular, for Borel $B \subseteq Y$,

$$w(B) = r(\nu')(B) = \int_{X_1} \mu_x(B)d\nu(x) - \nu(B). \tag{1.3}$$

Thus, we have also (since μ_x and ν have disjoint supports)

$$|w|(B) = \int_{X_1} \mu_x(B)d|\nu|(x) + |\nu|(B). \tag{1.3$'$}$$

Specializing to Borel $A \subseteq X_1$, we arrive at

$$w(A) = r(\nu')(A) = -\nu(A) = -\nu'(A') \tag{1.4}$$

the latter equality by (1.2) and

$$|w|(A) = |\nu|(A) = |\mu'|(A'). \tag{1.4$'$}$$

Since every Borel set in X_1' is of the form A' for some Borel A in X_1, we see that w determines ν' up to a possible atom at 0. Since we saw above that $r(\nu') = r(\mu')$, w determines uniquely a measure μ' on X' without any atom at 0 such that $r(\mu') = w$.

We claim that W is weak* closed. Indeed, it suffices to show that W is norm closed [3, V. 5.9]. Let $\{w_\alpha\}$ be a norm Cauchy net in W converging to $w \in C^*(Y)$. Then there are (unique) measures, without atoms at 0, μ_α' on X' such that $w_\alpha = r(\mu_\alpha')$. Hence,

$$\begin{aligned} \|\mu_\alpha' - \mu_\beta'\| &= |\mu_\alpha' - \mu_\beta'|(X_1') \\ &= |w_\alpha - w_\beta|(X_1) \text{ by (1.4)}' \\ &\leqq \|w_\alpha - w_\beta\| \end{aligned}$$

and so $\{\mu_\alpha'\}$ is norm Cauchy. Since X' is compact, there is a bounded regular Borel measure μ' on X' such that $\mu_\alpha' \to \mu'$. Hence $w = r(\mu')$ and so $w \in W$.

Obviously, the linear span of X' is dense in W so that $W = V^\perp$. We now claim that $W \neq C^*(Y)$. Indeed, choose $y \in Y - X_1$. If $\delta(y)$ were in W, it would determine a (unique) measure on X' with no atom at 0. Call it ν'. Then (1.4) implies that $0 = \delta(y)\ (A) = -\nu'(A')$ for each Borel set A' in X_1'. Thus $\nu' \equiv 0$ and so $\delta(y) = r(0) = 0$. This obvious contradiction shows that $W \neq C^*(Y)$, i.e. $V \neq \{0\}$.

The dual of V is $C^*(Y)/W$. We claim that each class of $C^*(Y)/W$ contains one and only one member of $C^*(Y; X_1)$. For let $m \in C^*(Y)/W$ and suppose that $\mu_1, \mu_2 \in m$ each were in $C^*(Y; X_1)$. Then $\mu_1 - \mu_2 = w \in W \cap C^*(Y; X_1)$. Then w determines a measure ν'on X' with no atom at 0. Since $w \in C^*\ (Y; X_1)$, $w(A) = 0$

for each Borel A in X_1 and so v' is the zero measure (by (1.4)). Hence, w is the zero measure and so $\mu_1 = \mu_2$. On the other hand, let $\mu \in m$. We associate to μ a measure $\bar{\mu}'$ on X_1' by restricting μ to X_1 and using the homeomorphism to lift it to X_1'. We extend $\bar{\mu}'$ to X' by giving it no atom at 0. Thus, if A' is a Borel set in X', letting $A = \{x \in X_1 \,|\, \mu_x - \delta(x) \in A'\}$, we have

$$\bar{\mu}'(A') \equiv \mu(A).$$

Since $\bar{\mu}'$ is a regular Borel measure, $r(\bar{\mu}') \in W$. By (1.4), for each Borel $A \subseteq X_1$,

$$(\mu + r(\bar{\mu}'))(A) = \mu(A) - \bar{\mu}'(A') = 0$$

so $\mu + r(\bar{\mu}') \in C^*(Y; X_1)$. As it also is in the class m, the claim is demonstrated. $\mu + r(\bar{\mu}')$ depends only on the class m and not on the particular representative μ. We then may define a map ϕ: $V^* \to C^*(Y; X_1)$ by

$$\phi(m) = \mu + r(\bar{\mu}')$$

for each $m \in V^*$ and any representative $\mu \in m$.

It is obvious that ϕ is a positive, linear, one-to-one mapping of V^* onto $C^*(Y; X_1)$. We need only show that ϕ is an isometry to complete the proof. Let $\mu \in m \in V^*$. Then

$$\begin{aligned} \|\phi(m)\| &= |\phi(m)|(Y) \\ &= |\phi(m)|(Y - X_1) \\ &= |\mu + r(\bar{\mu}')|(Y - X_1) \\ &\leqq |\mu|(Y - X_1) + |r(\bar{\mu}')|(Y - X_1) \end{aligned}$$

but then associating $\bar{\mu}$ with $\bar{\mu}'$ by (1.2) and using (1.3)$'$

$$= |\mu|(Y - X_1) + \int_{X_1} \mu_x(Y - X_1) d|\bar{\mu}|(x)$$

but obviously $\bar{\mu}$ agrees with μ on X_1

$$= |\mu|(Y - X_1) + \int_{X_1} \mu_x(Y - X_1) d|\mu|(x)$$

but then since $\mu_x(Y - X_1) = \|\mu_x\| \leqq 1$ for $x \in X_1$,

$$\begin{aligned} &\leqq |\mu|(Y - X_1) + |\mu|(X_1) \\ &= |\mu|(Y) \\ &= \|\mu\|. \end{aligned}$$

Since $\|m\| = \inf_{\mu \in m} \|\mu\|$, we have $\|m\| = \|\phi(m)\|$.

We note that unless X_1 is open, $C^*(Y - X_1)$ is not weak* closed in $C^*(Y)$. In fact, if $\{x_n\} \subseteq Y - X_1$ converges to $x \in X_1$, then $\{\delta(x_n)\} \subseteq C^*(Y; X_1)$ converges weak* to $\delta(x) \notin C^*(Y; X_1)$. In V^*, the class of $\delta(x_n)$ converges weak* to the class of μ_x which is in V^*. Thus, V^* is not isomorphic to $C^*(Y - X_1)$ in the weak* topology for X_1 not open.

A possible area of generalization is to relax the requirement that $\mu_x(X_1) = 0$ for each $x \in X_1$. The only result we have in that direction allows atoms at the points of X_1. As such measures are of doubtful importance, we shall not produce the long tedious proof and refer the reader to [9, Th. 5.19 and Cor. 5.22].

THEOREM 1.3. *Let Y be a compact metric space. Let*

$$X = \{x_j \mid j = 1,2,\cdots\} \bigcup \{z\}$$

be a set of distinct points in Y such that $\lim x_j = z$. Let $\{\mu_j \mid j = 1,2,\cdots\}$ and ν be positive measures on Y, with norms bounded by one. Suppose these measures satisfy the following:

If k is a bounded function on Y which is a pointwise limit of a a sequence of continuous functions, then

$$\lim \mu_j(k) = \nu(k).$$

Suppose $Y \neq X$. Then

$$V = \{f \in C(Y) \mid f(z) = \nu(f), f(x_j) = \mu_j(f), j = 1,2,\cdots\}$$

with the relative norm and order is a non-trivial separable simplex space.

THEOREM 1.4. *Let Y be a compact Hausdorff space and $x_1,\cdots,x_N$ be distinct points not exhausting Y. Let $\mu_1,\cdots,\mu_N$ be positive measures with norms bounded by one. Then*

$$V = \{f \in C(Y) \mid f(x_j) = \mu_j(f),\ j = 1,\cdots,N\}$$

with the relative norm and order is a non-trivial simplex space.

We should like to point out that if

$$\sum_{i=1}^{N} \mu_j(\{x_i\}) \neq 1 \text{ for } j = 1,\cdots,N,$$

then we may fashion a proof of Theorem 1.4 by inverting a system of N linear equations and then applying Theorem 1.2. We note that [1, Prop. 3] gives the above result for $\|\mu_i\| = 1$ for each $i = 1,\cdots,N$. (We thank the referee for this reference.) Also, the case for $N = 1$ was previously established by F. Perdrizet.

A second possible area of generalization is to relax the requirement that the map $x \to \mu_x$ is weak* continuous. In this vein, we offer the following mild generalization.

THEOREM 1.5. *Let Y be a compact Hausdorff space. Let*

$$X = \{x_j \mid j = 1,2, \cdots\} \bigcup \{z\}$$

be a set of distinct points in Y such that $\lim x_j = z$. Let $\{\mu_j \mid j = 1,2,\cdots\}$ and λ be positive measures on Y, with norms bounded by one. We assume:

1) $\mu_j(X) = 0 = \lambda(X) \qquad j = 1,2,\cdots$
2) $X \neq Y$
3) $\lim \mu_j = \delta(z)$ *in the weak* topology.*

Then

$$V = \{f \in C(Y) \mid f(z) = \lambda(f),\, f(x_j) = \mu_j(f), j = 1,\cdots\}$$

is a non-trivial simplex space with the relative order and norm. Further, V^* is isometrically order isomorphic to $C^*(Y; X)$.

PROOF. The proof is virtually identical to that of Theorem 1.2 and we will only indicate the changes. First,

$$X' = \{\mu_j - \delta(x_j) \mid j = 1,2,\cdots\} \bigcup \{\lambda - \delta(z)\}$$

is not compact but

$$X'' = X' \bigcup \{0\}$$

is. Also, $\{x_j\}$ is homeomorphic to $\{\mu_j - \delta(x_j)\}$ and W hasthe same form as before. Let $w \in W$ and $w = r(v'')$ with v'' a measure on X''. Remove the possible atom at 0 from v''; call the resulting measure μ''. Noting the possible atom of μ'' at the point $\lambda - \delta(z)$, we get for each Borel $B \subseteq Y$

$$(1.5) \quad w(B) = \Sigma \mu''(\{\mu_j - \delta(x_j)\})[\mu_j(B) - \delta(x_j)(B)] + \mu''(\{\lambda - \delta(z)\}[\lambda(B) - \delta(z)(B)]$$

instead of (1.3). Using (1.5), we see that each $w \in W$ determines uniquely a measure μ'' on X'' such that $w = r(\mu'')$ and μ'' has no atom at 0. Then W is closed and $W \neq C^*(Y)$ as before. In the identification of $V^* = C^*(Y)/W$, the only change is in definition of $\bar{\mu}''$ given a measure μ on Y. We define

$$\bar{\mu}''(\{\lambda - \delta(z)\}) \equiv \mu(\{z\})$$

so $\bar{\mu}''$ is now a measure on X''. The remainder of the proof is as before.

2. Examples

EXAMPLE 2.1. If $\max V$ is compact, $\max V$ still may not be locally compact. This holds even if V is a GC-space [6].

We take V to be

$$V = \{f \in C[0,1] \,|\, f\left(\frac{1}{n}\right) = \frac{1}{2}\left(f(1) + \frac{1}{n} f\left(\frac{2}{3}\right)\right) \qquad n = 2,\ 3, \cdots$$

$$f(0) = \frac{1}{2} f(1)\}.$$

In the notation used in Theorem 1.2, we have $X = \{0\} \bigcup \{1/n \,|\, n = 2, 3, \cdots\}$, $\mu_{1/n} = 1/2(\delta(1) + (1/n)\delta(2/3))$ and $\mu_0 = 1/2\delta(1)$. As $\mu_{1/n} \to \mu_0$ in norm, V is a non-trivial separable simplex space with

$$EP_1(V) - \{0\} = \{\delta(x) \,|\, 0 < x \leqq 1 \text{ and } x \neq \frac{1}{n},\ n = 2, 3, \cdots\}.$$

As

$$Z = \{\delta(x) \,|\, 0 \leqq x \leqq 1\}$$

the zero measure is not in Z and so $\max V$ is compact [7, Th. 3.2]. Since

$$\bigcup \{\Phi(x) \,|\, x \in Z - EP_1(V)\} = \left\{\delta(1), \delta\left(\frac{2}{3}\right)\right\}$$

is finite, V is a GC-space [8, Cor. 9]. Now consider $\psi(\delta(2/3))$. $\delta(1/n) \in \psi(\delta(2/3))$ for $n = 2, 3, \cdots$, but $\delta(0) \notin \psi(\delta(2/3))$. Since

$$\lim \delta\left(\frac{1}{n}\right) = \delta(0),$$

$\psi(\delta(2/3))$ is not closed. Hence, $\max V$ is not locally compact [7, Th. 2.5 and Th. 3.3].

EXAMPLE 2.2. (For the notation and definitions see [6].) We say that a simplex property P is *absolute* if closed ideals are P-ideals iff they are themselves P-spaces. If this is not the case, we say that P is *relative*. The property C is absolute (from its very definition) and the property M is relative [6, 7.1]. We shall now present an example to show that GM is a relative property. In fact, we will construct a separable NM space V with an ideal I such that I is an M-space in itself.

Let $\{c_t\}$ be a sequence of real numbers, bounded above by one, monotonically decreasing to zero. Let ϕ be a one-to-one map of N, the natural numbers, onto $N \times N$ such that

(2.1) $\phi(t) = (m,n)$ where if t is odd (even), then m is odd (even).

Let N^* be the one-point compactification of N. Let V be

$$V = \{f \in C(N^* \times N^*) \,|\, f(\infty,j) = 0$$

$$(2.2) \qquad f(t,\infty) = \frac{1}{3}c_t(f(m,n) + f(m+1,n)) \quad t \text{ odd}$$

$$f(t,\infty) = \frac{1}{3}c_t(f(m,n) + f(m,n+1)) \quad t \text{ even}$$

$$\text{where} \quad \phi(t) = (m,n)\}.$$

In the notation of Theorem 1.2,

$$X = \{(t,\infty) \,|\, t = 1,\cdots\} \bigcup \{(\infty,t) \,|\, t = 1,\cdots\} \bigcup \{(\infty,\infty)\}.$$

Then $\mu(t\,\infty)$ is $1/3\, c_t(\delta, (m,n) + \delta(m+1,n))$ or $1/3\, c_t(\delta(m,n) + \delta(m,n+1))$ (see (2.2)) and $\mu(\infty,t) = \mu(\infty,\infty) = 0$. As

$$\| \mu(t,\infty) \| \leqq \frac{2}{3}c_t,$$

$x \to \mu_x$ is norm continuous. Hence V is a separable non-trivial simplex space with

$$EP_1(V) - \{0\} = \{\delta(a,b) \,|\, a \text{ and } b \text{ finite}\}.$$

Let I be the closed ideal which is defined by

$$I = \{f \in V / f(i,j) = 0 \text{ if } i \text{ is even}\}.$$

Then using (2.1) and (2.2),

$$I = \{f \in C(N^* \times N^*) \,|\, f(\infty,j) = 0$$

$$f(i,j) = 0 \text{ if } i \text{ is even}$$

$$f(t,\infty) = \frac{1}{3}c_t f(m,n) \text{ for } t \text{ odd}$$

$$f(t,\infty) = 0 \text{ for } t \text{ even}$$

$$\text{where} \quad \phi(t) = (m,n)\}.$$

Since $Z(I)$ lies in the extremal rays of $P_1(I)$, I is an M-space in its own right. We claim that V is NM. Indeed, it follows from (2.2) that, letting $E^+ = EP_1(V) - \{0\}$,

$$d_1(E^+) = e_1(E^+) = E^+.$$

But then [8, Prop. 5] implies that V is NM.

ACKNOWLEDGMENTS

Parts of this paper are from my Ph.D. dissertation submitted to Stanford University, written under the direction of Professor E. Effros of the University of Pennsylvania. I would like to thank the referee for his very illuminating comments and Professor M. M. Rao for many useful discussions.

REFERENCES

1. Bednar, J. and H. Lacey, *On spaces whose duals are abstract L-spaces and Choquet Simplexes*, to appear.
2. Bourbaki, N., *General Topology*, part 1, Addison-Wesley, Reading, 1966.
3. Dunford, N. and J. Schwartz, *Linear Operators*, part 1, Interscience, New York, 1964.
4. Effros, E., *Structure in Simplexes*, Acta Math. **117** (1967), 103-121.
5. Effros, E., *Structure in Simplexes II*, J. Func. Anal. **1** (1967), 379-391.
6. Effros, E., and A. Gleit, *Structure in Simplexes III*, Trans. Amer. Math. Soc. **142** (1969), 355-379.
7. Gleit, A., *On the structure topology of simplex spaces*, Pacific J. Math. **34** (1970), 389–406.
8. Gleit, A., *Composition series for simplex spaces*, to appear.
9. Gleit, A., *Topics in simplex space theory*, Ph.D. thesis, Stanford, 1969.
10. Bourbaki, N., *Integration*, N. 1244, Hermann, Paris, 1967.
11. Phelps, R., *Lectures on Choquet's Theorem*, van Nostrand, Princeton, 1966.
12. Taylor, P., *The structure space of a Choquet simplex*, J. Func. Anal. **6** (1970), 208–217.

DEPARTMENT OF MATHEMATICS
CARNEGIE MELLON UNIVERSITY
PITTSBURGH, PENN.